Defined functions:

$$f(m,n) = \begin{cases} n & m = 0 \\ 0 & n = 0 \\ n \cdot f(m-1,n) - g(m-1,n-1) & \text{otherwise} \end{cases}$$

$$g(x,y) = \begin{cases} 0 & y = 0 \\ f(x,y) + g(x,y-1) & \text{otherwise} \end{cases}$$

Lemma 1: $g(x,y) = \sum_{i=1}^{y} f(x,i)$
Base case ($y = 0$): $g(x,0) = 0 = \sum_{i=1}^{0} f(x,i)$ (defn of g, f)
Induction: Assume for some $k \geq 0$, $g(x,k) = \sum_{i=1}^{k} f(x,i)$.

$$\begin{aligned} g(x,k+1) &= f(x,k+1) + g(x,k) && \text{(defn of g)} \\ &= f(x,k+1) + \textstyle\sum_{i=1}^{k} f(x,i) && \text{(induction hypothesis)} \\ &= \textstyle\sum_{i=1}^{k+1} f(x,i) && \text{(defn of summation)} \end{aligned}$$

□

Lemma 2: $\sum_{i=1}^{n-1} \sum_{j=1}^{i} i^m = \sum_{j=1}^{n} j^m (n-j)$
Base case ($n = 0$): $\sum_{i=1}^{-1} \sum_{j=1}^{i} i^m = 0 = \sum_{j=1}^{0} j^m (n-j)$
Induction: Assume for some $k \geq 0$, $\sum_{i=1}^{k-1} \sum_{j=1}^{i} i^m = \sum_{j=1}^{k} j^m (k-j)$.

$$\begin{aligned} \textstyle\sum_{i=1}^{k} \sum_{j=1}^{i} i^m &= \textstyle\sum_{i=1}^{k-1} \sum_{j=1}^{i} i^m + \sum_{j=1}^{k} j^m && \text{(defn of sum)} \\ &= \textstyle\sum_{i=1}^{k} i^m (k-i) + \sum_{j=1}^{k} j^m && \text{(induction hypothesis)} \\ &= \textstyle\sum_{i=1}^{k} i^m (k-i) + i^m && \text{(summation property)} \\ &= \textstyle\sum_{i=1}^{k} i^m (k-i+1) && \text{(collection of terms)} \\ &= \textstyle\sum_{i=1}^{k+1} i^m (k-i+1) && \text{(summation property)} \end{aligned}$$

□

Theorem: $f(m,n) = \sum_{i=1}^{n} i^m$
Base case ($m = 0$): $f(0,n) = n = \sum_{i=1}^{n} i^0$ (defn of f)
Induction: Assume for some $k \geq 0$, $f(k,n) = \sum_{i=1}^{n} i^k$.

$$\begin{aligned} f(k+1,n) &= n \cdot f(k,n) - g(k,n-1) && \text{(defn of f)} \\ &= n \cdot \textstyle\sum_{i=1}^{n} (i^k) - g(k,n-1) && \text{(induction hypothesis)} \\ &= n \cdot \textstyle\sum_{i=1}^{n} (i^k) - \sum_{j=1}^{n-1} f(k,j) && \text{(Lemma 1)} \\ &= n \cdot \textstyle\sum_{i=1}^{n} (i^k) - \sum_{j=1}^{n-1} \sum_{c=1}^{j} c^k && \text{(induction hypothesis)} \\ &= n \cdot \textstyle\sum_{i=1}^{n} (i^k) - \sum_{j=1}^{n} (n-j) j^k && \text{(Lemma 2)} \\ &= \textstyle\sum_{i=1}^{n} (n i^k - (n-i) i^k) && \text{(summation property)} \\ &= \textstyle\sum_{i=1}^{n} i^{k+1} && \text{(collection of terms)} \end{aligned}$$

□